第３章 8 「極座標による２重積分（発展その２）」 第２回

解答

1. (1) $\dfrac{\pi}{8}$

(2) $\dfrac{\pi}{3}$

(3) $\dfrac{16}{5}\pi$

解説

1. (1) $\displaystyle\iint_D x^2dxdy$

$$
\begin{aligned}
&= \int_0^{\pi}\int_0^1 (r\cos\theta)^2\,rdrd\theta \\
&= \int_0^{\pi}\left\{\int_0^1 r^3dr\right\}\cos^2\theta\,d\theta \\
&= \int_0^{\pi}\left[\frac{1}{4}r^4\right]_0^1\cos^2\theta\,d\theta \\
&= \frac{1}{4}\int_0^{\pi}\cos^2\theta\,d\theta \\
&= \frac{1}{4}\int_0^{\pi}\frac{1+\cos2\theta}{2}\,d\theta \\
&= \frac{1}{8}\int_0^{\pi}(1+\cos2\theta)\,d\theta \\
&= \frac{1}{8}\left[\theta+\frac{1}{2}\sin2\theta\right]_0^{\pi} \\
&= \frac{\pi}{8}
\end{aligned}
$$

(2) $\displaystyle\iint_D x^2\left(x^2+y^2\right)dxdy$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}} (r\cos\theta)^2\,r^2rdrd\theta \\
&= \int_0^{\frac{\pi}{2}}\left\{\int_0^{\sqrt{2}} r^5dr\right\}\cos^2\theta\,d\theta \\
&= \int_0^{\frac{\pi}{2}}\left[\frac{1}{6}r^6\right]_0^{\sqrt{2}}\cos^2\theta\,d\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}}\frac{1+\cos2\theta}{2}\,d\theta \\
&= \frac{2}{3}\int_0^{\frac{\pi}{2}}(1+\cos2\theta)\,d\theta \\
&= \frac{2}{3}\left[\theta+\frac{1}{2}\sin2\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{3}
\end{aligned}
$$

(3) $\displaystyle\iint_D y^2\sqrt{x^2+y^2}dxdy$

$$
\begin{aligned}
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 (r\sin\theta)^2\sqrt{r^2}rdrd\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\int_0^2 r^4dr\right\}\sin^2\theta\,d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{1}{5}r^5\right]_0^2\sin^2\theta\,d\theta \\
&= \frac{32}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\,d\theta \\
&= \frac{32}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-\cos2\theta}{2}\,d\theta \\
&= \frac{16}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos2\theta)\,d\theta \\
&= \frac{16}{5}\left[\theta-\frac{1}{2}\sin2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{16}{5}\pi
\end{aligned}
$$