第３章 7 「極座標による２重積分（発展その１）」 第２回

解答

1. (1) 8

(2) $\dfrac{1}{24}$

(3) $\dfrac{1}{18}$

(4) $\dfrac{1}{4}$

解説

1. (1) $\displaystyle\iint_D xydxdy$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \int_0^{2\sqrt{2}} r\cos\theta\, r\sin\theta\, rdrd\theta \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_0^{2\sqrt{2}} r^3 dr \right\} \cos\theta\, \sin\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4} r^4 \right]_0^{2\sqrt{2}} \cos\theta\, \sin\theta\, d\theta \\
&= 16 \int_0^{\frac{\pi}{2}} \cos\theta\, \sin\theta\, d\theta \\
&= 16 \left[\frac{1}{2} \sin^2\theta \right]_0^{\frac{\pi}{2}} \\
&= 8
\end{aligned}$$

(2) $\displaystyle\iint_D xy^3dxdy$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \int_0^{1} r\cos\theta\, (r\sin\theta)^3\, rdrd\theta \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_0^{1} r^5 dr \right\} \cos\theta\, \sin^3\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{6} r^6 \right]_0^{1} \cos\theta\, \sin^3\theta\, d\theta \\
&= \frac{1}{6} \int_0^{\frac{\pi}{2}} \cos\theta\, \sin^3\theta\, d\theta \\
&= \frac{1}{6} \left[\frac{1}{4} \sin^4\theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{24}
\end{aligned}$$

(3) $\displaystyle\iint_D x^2y\sqrt{x^2+y^2}dxdy$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \int_0^{1} (r\cos\theta)^2\, r\sin\theta\, \sqrt{r^2}\, rdrd\theta \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_0^{1} r^5 dr \right\} \cos^2\theta\, \sin\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{6} r^6 \right]_0^{1} \cos^2\theta\, \sin\theta\, d\theta \\
&= \frac{1}{6} \int_0^{\frac{\pi}{2}} \cos^2\theta\, \sin\theta\, d\theta \\
&= \frac{1}{6} \left[-\frac{1}{3} \cos^3\theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{18}
\end{aligned}$$

(4) $\displaystyle\iint_D \frac{xy}{x^2+y^2}dxdy$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} \frac{r\cos\theta\, r\sin\theta}{r^2}\, rdrd\theta \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_1^{\sqrt{2}} rdr \right\} \cos\theta\, \sin\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} r^2 \right]_1^{\sqrt{2}} \cos\theta\, \sin\theta\, d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos\theta\, \sin\theta\, d\theta \\
&= \frac{1}{2} \left[\frac{1}{2} \sin^2\theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{4}
\end{aligned}$$