第3章 7 「極座標による2重積分（発展その1)」 第3回

解答

1. (1) $\dfrac{1}{35}$

(2) $\dfrac{15}{8}$

(3) $\dfrac{1}{27}$

(4) $\dfrac{1}{2}\log 2$

解説

1. (1)
$$\begin{aligned}
&\iint_D x^4 y\,dxdy \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 (r\cos\theta)^4 \; r\sin\theta\, r\,dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_0^1 r^6 dr \right\} \cos^4\theta \,\sin\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{7} r^7 \right]_0^1 \cos^4\theta \,\sin\theta\, d\theta \\
&= \frac{1}{7} \int_0^{\frac{\pi}{2}} \cos^4\theta \,\sin\theta\, d\theta \\
&= \frac{1}{7} \left[-\frac{1}{5}\cos^5\theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{35}
\end{aligned}$$

(2)
$$\begin{aligned}
&\iint_D xy\,dxdy \\
&= \int_0^{\frac{\pi}{2}} \int_1^2 r\cos\theta\, r\sin\theta\, r\,dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_1^2 r^3 dr \right\} \cos\theta \,\sin\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{4} r^4 \right]_1^2 \cos\theta \,\sin\theta\, d\theta \\
&= \frac{15}{4} \int_0^{\frac{\pi}{2}} \cos\theta \,\sin\theta\, d\theta \\
&= \frac{15}{4} \left[\frac{1}{2}\sin^2\theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{15}{8}
\end{aligned}$$

(3)
$$\begin{aligned}
&\iint_D xy^2 \left(x^2+y^2\right)^2 dxdy \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 r\cos\theta \,(r\sin\theta)^2 \; (r^2)^2\, r\,dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_0^1 r^8 dr \right\} \cos\theta \,\sin^2\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{9} r^9 \right]_0^1 \cos\theta \,\sin^2\theta\, d\theta \\
&= \frac{1}{9} \int_0^{\frac{\pi}{2}} \cos\theta \,\sin^2\theta\, d\theta \\
&= \frac{1}{9} \left[\frac{1}{3}\sin^3\theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{27}
\end{aligned}$$

(4)
$$\begin{aligned}
&\iint_D \frac{xy}{\left(x^2+y^2\right)^2} dxdy \\
&= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{r\cos\theta\, r\sin\theta}{(r^2)^2}\, r\,dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \left\{ \int_1^2 \frac{1}{r} dr \right\} \cos\theta \,\sin\theta\, d\theta \\
&= \int_0^{\frac{\pi}{2}} \Big[\log|r| \Big]_1^2 \cos\theta \,\sin\theta\, d\theta \\
&= \log 2 \int_0^{\frac{\pi}{2}} \cos\theta \,\sin\theta\, d\theta \\
&= \log 2 \left[\frac{1}{2}\sin^2\theta \right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}\log 2
\end{aligned}$$